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Covariant Curves of the Plane Rational Quintic.

BY J. I. TRACEY.

Introduction.

Let a plane rational curve ρ^n of order n be given parametrically by the equations

$$x_i = (\alpha_i t)^n \qquad (i=0, 1, 2), \tag{1}$$

then any line of the plane intersects ρ^n in n points whose equation may be represented symbolically by the binary n-ic.

$$(a\xi)(\alpha t)^n = 0. \tag{2}$$

We shall designate this form by $C_{1,n}$, since it is of the n-th order in the parameter t and involves the coefficients of the line ξ to the first degree.

If I_k is an invariant of $C_{1,n}$, it contains the coefficients of the line to the k-th degree and, for a varying ξ, represents a curve of class k. This curve is the locus of line sections of ρ^n for which I_k vanishes.

Any covariant $C_{l,m}$ of $C_{1,n}$ is of degree l in the ξ's and of order m in the t's. Now for a given ξ we obtain m points $\tau_1, \tau_2, \ldots, \tau_m$ on ρ^n. These are the roots of the equation $C_{l,m}=0$. However, for a given point of ρ^n, say $t=\tau$, $C_{l,m}$ becomes a curve of class l which is the locus of line sections of ρ^n containing τ for a root of its $C_{l,m}$.

If for a given line $C_{l,m}$ vanishes identically, then the coefficients of this m-ic give a linear system of at most $m+1$ curves of class l, each containing all line sections of ρ^n for which $C_{l,m}$ vanishes identically. By this means we can frequently express in concise form all the independent curves of a given class which contain special line sections of ρ^n, and account for all degenerate curves whose degeneracy is due to the multiplicity of the system.

It is important to keep in mind that when a covariant $C_{l,m}$ vanishes identically, all invariants and covariants of $C_{l,m}$ likewise vanish. Interpreted geometrically, if a linear system of curves given by $C_{l,m}$ contains a set of lines L, then any curve or system of curves derived from invariants and other covariants of $C_{l,m}$, respectively, will contain lines L and contain them multiply.

Thus, if I_{ql} is an invariant of $C_{l,m}$, each line L is a q-fold line of the curve represented by this invariant. A covariant of $C_{l,m}$ containing its coefficients to the r-th degree will determine a linear system of curves of class rl, and L will be r-fold lines of this system. Finally, any operation of $C_{l,m}$ on a second form, or conversely, will give a system of curves on L.

§ 1. *Covariant Curves on the Double Lines and Stationary Lines of ρ^n.*

The curve ρ^n has in general $3(n-2)$ stationary lines and $2(n-2)(n-3)$ double lines. Now there is a covariant $C_{2(n-2),2(n-2)}$ which vanishes when $C_{1,n}$ has a triple root, and also vanishes if $C_{1,n}$ contains two double roots. This covariant gives the $2n-3$ independent curves of class $2(n-2)$ on the $(n-2)(2n-3)$ stationary lines and double lines of ρ^n. In terms of the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $C_{1,n}$ it is

$$C_{2(n-2),2(n-2)} \equiv \sum^{n} (\alpha_1-\alpha_2)^2 (\alpha_1-\alpha_3)^2 \ldots (\alpha_1-\alpha_{n-1})^2 (\alpha_2-\alpha_3)^2 \ldots (\alpha_{n-2}-\alpha_{n-1})^2 (t-\alpha_n)^{2(n-2)}. \qquad (3)$$

For the rational cubic this covariant is the $C_{2,2}$,* the Hessian of the $C_{1,3}$, and its coefficients give three independent conics on the three stationary lines.

In the case of the ρ^4 the corresponding covariant is †

$$C_{4,4} \equiv 3I_3C_{1,4} - 2I_2C_{2,4}, \qquad (4)$$

and its coefficients equated to zero give five independent quartic curves on the six stationary lines and four double lines of the base curve.

Similarly for the ρ^5 the covariant is ‡

$$C_{6,6} \equiv 4C_{2,2}^3 + 12C_{2,2}C_{4,4} + C_{2,6}I_4 - 216C_{3,3}^2. \qquad (5)$$

From the coefficients of this covariant we obtain a linear system of seven independent sextics, by means of which the six-fold infinity of sextics on the nine stationary lines and twelve double lines of ρ^5 may be expressed.

§ 2. *Covariant Curves on the Double Lines of ρ^n.*

The condition for $C_{1,n}$ to have two double roots is the identical vanishing of a $C_{2n-5,3(n-2)}$. The coefficients of this covarient give $3n-5$ curves of class $2n-5$ on the $2(n-2)(n-3)$ double lines of ρ^n. Now since there are only $3(n-2)$ linearly independent curves of class $2n-5$ on $2(n-2)(n-3)$ fixed lines, we shall expect the existence of a linear relation among these $3n-5$ coefficients. Expressed in terms of the roots of $C_{1,n}$ this covariant is

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* The fundamental forms used throughout are, unless otherwise stated, the same as given by Salmon, "Higher Algebra," fourth edition, or by Elliott, "Algebra of Quantics." The notation is the same as Elliott's.

† Salmon, "Higher Algebra," p. 199.

‡ This covariant corresponds in form to the one given by Cayley, *Collected Mathematical Papers*, Vol. IV, p. 276.

$$C_{2n-5,3(n-2)} \equiv \sum^{\frac{n(n-1)(n-2)}{6}} (\alpha_1-\alpha_4)\ldots(\alpha_1-\alpha_n)(\alpha_2-\alpha_4)\ldots \\ (\alpha_2-\alpha_n)(\alpha_3-\alpha_4)\ldots(\alpha_3-\alpha_n)(\alpha_4-\alpha_5)^2\ldots \\ (\alpha_{n-1}-\alpha_n)^2(t-\alpha_1)^{n-2}(t-\alpha_2)^{n-2}(t-\alpha_3)^{n-2}. \qquad (6)$$

For the ρ^4 this covariant is the $C_{3,6}$ giving seven cubic curves on the four double lines. It has been proven[*] that this form is apolar to the sextic equation whose roots give the six points of inflection of the ρ^4, thus accounting for the linear relation which has been mentioned.

The condition that $C_{1,5}$ have two double roots is the vanishing of the covariant[†]

$$C_{5,9} \equiv 50C_{2,6}C_{3,3}-C_{1,5}(C_{4,4}+3C_{2,2}^2). \qquad (7)$$

These coefficients give ten curves of class 5 on the twelve double lines of the ρ^5, and the apolarity relation existing between this covariant and the equation which gives the nine points of inflection has been proven by direct calculation.

The covariant whose vanishing is the condition that $C_{1,6}$ have two double roots is as follows:[‡]

$$C_{7,12} \equiv 36C_{3,12}I_2^2-35C_{4,6}C_{1,6}I_2+25[19C_{2,4}C_{5,8}-2C_{2,4}C_{3,8}I_2 \\ +21C_{3,2}C_{4,10}-18C_{3,6}C_{4,6}+C_{3,8}C_{4,4}], \qquad (8)$$

and in a similar manner as above these coefficients give septimic curves on the twenty-four double lines of the rational sextic. That the apolarity relation exists between the $C_{7,12}$ and the equation giving the twelve points of inflection of ρ^6 is due to the following general theorem. We shall now prove that:

The covariant $C_{2n-5,3(n-2)}$, *whose vanishing is the condition for two double roots, is apolar to the* $3(n-2)$*-ic which gives the points of inflection of* ρ^n.

Each curve of class $2n-5$ determined by the coefficients of the covariant has $2(3n-7)$ common lines with the ρ^n in addition to the double lines. Now the base curve ρ^n may be represented in point form by a function of $\{A, \xi, t^n\}$,[§] and in line form by $\{A^2, x, t^{2(n-1)}\}$. The $C_{2n-5,3(n-2)}$ is similarly represented by $\{A^{2n-5}, \xi^{2n-5}, t^{3(n-2)}\}$, and on writing its apolarity relation with an arbitrary form $(\alpha t)^{3(n-2)}$ the resulting expression is $\{A^{2n-5}, \xi^{2n-5}, \alpha\}$. This represents a curve of class $2n-5$ on the double lines of ρ^n. The contacts of common lines to these two curves are given by the incidence condition of ξ and x in the forms $\{A^{2n-5}, \xi^{2n-5}, \alpha\}$ and $\{A^2, x, t^{2(n-1)}\}$, respectively, the result being the equation

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[*] Thomsen, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, p. 230.

[†] Cayley, *Collected Mathematical Papers*, Vol. II, p. 469. Cayley's $C_{4,4}$ is Elliott's $C_{2,2}^2-C_{4,4}$.

[‡] The fundamental forms of $C_{1,6}$ used in the calculation of this covariant were taken from Cayley, *Collected Mathematical Papers*, Vol. XI, p. 377.

[§] This is another way of writing $(a\xi)(at)^n$, the parametric equation of ρ^n, A representing the coefficients of the binary n-ics cut out by the reference triangle. The exponent of any quantity indicates the degree to which that quantity enters in the given expression, the brackets, $\{\ \}$, meaning a function of the enclosed quantities.

$$\{A^{3(2n-5)}, \alpha, t^{2(n-1)(2n-5)}\} = 0. \tag{9}$$

Equation (9) contains, as we have already seen, the contacts of the double lines which involve A and t to the degrees $6(n-3)$ and $4(n-2)(n-3)$, respectively. On removing this factor we are left with the contacts of the extra common lines which are given by the equation

$$\{A^3, \alpha, t^{2(3n-7)}\} = 0. \tag{10}$$

Now it must be possible to obtain equation (10) by means of the transvectant process, so we shall examine the covariant sets of points along ρ^n.

There is no such covariant which involves the A's to a degree less than 3, and the highest order of such in the variable is $3(n-2)$. This of course is the form giving the points of inflection of ρ^n, which in this notation is $\{A^3, t^{3(n-2)}\}$. Therefore equation (10) is the first transvectant or the Jacobian of the arbitrary form $(\alpha t)^{3(n-2)}$ and the linear covariant $\{A^3, t^{3(n-2)}\}$.

In general, then, for every set of points given by $(\alpha t)^{3(n-2)}$ there is a curve of class $2n-5$ on the double lines formed by its apolarity relation with $C_{2n-5, 3(n-2)}$. However, if $(\alpha t)^{3(n-2)} \equiv \{A^3, t^{3(n-2)}\}$ their Jacobian vanishes and there is no such curve corresponding to this set of points. This proves the theorem.

§ 3. *Covariant Curves on the Stationary Lines.*

The covariant which vanishes when $C_{1,n}$ has a triple root is a $C_{2(n-3), 4(n-4)}$. From this form there are $4n-15$ curves of class $2(n-3)$ on the $3(n-2)$ stationary lines. There are, however, $2(n-2)(n-4)$ curves of class $2(n-3)$ on $3(n-2)$ lines, and for values of $n \overline{<} 6$, we shall show how all the remaining curves are given by invariants which vanish when $C_{1,n}$ has a triple root.

In terms of the roots of $C_{1,n}$ this covariant is

$$C_{2(n-3), 4(n-4)} \equiv \sum^{\frac{n(n-1)}{2}} (\alpha_1-\alpha_2)^2 \dots (\alpha_1-\alpha_{n-2})^2 (\alpha_2-\alpha_3)^2 \dots (\alpha_{n-3}-\alpha_{n-2})^2 \\ (\alpha_{n-1}-\alpha_n)^2 (t-\alpha_{n-1})^{2(n-4)} (t-\alpha_n)^{2(n-4)}. \tag{11}$$

When $n=4$ this becomes the invariant I_2 of the $C_{1,4}$, and it is well known that this conic touches the six stationary lines of ρ^4.

The condition that $C_{1,5}$ have a triple root is the vanishing of *

$$C_{4,4} \equiv 3C^1_{4,4} - C^2_{2,2}. \tag{12}$$

The invariant I_4 also vanishes for a triple root; hence the $C_{4,4}$ and the I_4 give the linear system of six independent quartic curves on the nine stationary lines of ρ^5.

The covariant in question becomes a $C_{6,8}$ for the $C_{1\,6}$, and this gives nine sextic curves on the twelve stationary lines. However, there are two invariants, an I_4 and an I_6, which vanish when $C_{1,6}$ has a triple root.† The quartic curve given by the I_4, together with any conic in the plane, is a degenerate sextic, and hence a five-fold infinity of the required system is given by this invariant. We have thus located sixteen independent sextic curves from which the entire linear system on the twelve stationary lines of the ρ^6 is determined.

§ 4. *Special Line Sections of ρ^5.*

To examine the curves given by the covariants and invariants of $C_{1,5}$ and study their relation to the base curve, we shall enumerate certain line sections of ρ^5, together with the reduced form which the corresponding binary quintics can be made to assume.

α. There are six so-called cyclic lines whose intersections with ρ^5 are given by a binary quintic which is apolar to a unique quadratic. Such line sections are reducible to the form x^5+y^5.

β. Fifteen lines such that four of the intersections are two harmonic pairs, while the remaining intersection is a root of the Jacobian of the two pairs. This form is reducible to $x(x^4+y^4)$.

γ. There are fourteen lines of ρ^5 such that the point of contact and three remaining intersections are a self-apolar quartic. The corresponding binary form may be written $x^2(x^3+y^3)$.

δ. Ten lines whose intersections with ρ^5 constitute a binary cubic and its Hessian. These can take the form $xy(x^3+y^3)$.

ε. There are nine stationary lines giving forms reducible to $x^3(x^2+y^2)$.

ξ. Twenty-four lines of ρ^5 such that the contact of each forms with the three remaining intersections a binary quartic which breaks up into harmonic pairs. These binary quintics can take the form $x^2y(x^2+y^2)$.

η. Twelve double lines which can be reduced to the form $x^2y^2(x+y)$.

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* Cayley, *Selected Mathematical Papers*, Vol. II, p. 469. The $C^1_{4,4}$ is Elliott's $C_{4,4}$.

† Salmon, "Higher Algebra," p. 263.

It is observed that the $C_{1,5}$ in each of these reduced forms consists of only two terms.

§ 5. *The First Osculants of* ρ^5.

Let the rational quintic be given by

$$x_i = a_i t^5 + 5b_i t^4 + 10c_i t^3 + 10d_i t^2 + 5e_i t + f_i \equiv (\alpha_i t)^5, \tag{13}$$

where $i=0, 1, 2$. Then the osculant quartic of a point τ on the base curve is obtained by polarizing equations (13) with respect to τ. It is

$$x_i \equiv (\alpha_i \tau)(\alpha_i t)^4. \tag{14}$$

Now on forming the conic g_2 of this quartic osculant we see at once that it is the $C_{2,2}$ of a line section when $t=\tau$. Therefore:

The locus of lines cutting out self-apolar quartics from the osculant curve (14) *is the conic* $C_{2,2}$ *formed for the given point* τ.

Similarly if we form the g_3 of a line section of (14), it is the canonizant of $C_{1,5}$ when $t=\tau$. Hence:

The locus of lines cutting harmonic pairs from the osculant quartic is the cubic curve $C_{3,3}$ *formed for the point* τ.

And from these we have immediately:

The sextic $C_{2,2}^3 - 27C_{3,3}^2$, *for any point* τ *of* ρ^5, *gives the line equation of the quartic osculant at that point.*

The osculant cubic at the point τ is

$$x_i = (\alpha_i \tau)^2 (\alpha_i t)^3, \tag{15}$$

and on taking the discriminant of a line section of this curve we find that:

$C_{4,8} \equiv C_{2,2} C_{2,6} - C_{1,5} C_{3,3}$, *for a point* τ, *gives the line equation of the corresponding cubic osculant.*

Polarizing $(\alpha_i t)^5$ three times with respect to τ we obtain the osculant conic, namely:

$$x_i = (\alpha_i \tau)^3 (\alpha_i t)^2, \tag{16}$$

and it is readily seen that:

The Hessian $C_{2,6}$ *formed for* τ *is the line equation of the osculant conic.*

§ 6. *Other Covariant Curves.*

Taking up the curves given by the fundamental covariants of $C_{1,5}$, we shall consider first the system of conics given by the $C_{2,2}$. Every point τ of the base curve determines a conic, and for every line ξ of the plane there are two definite points t_1 and t_2. Hence there are four lines associated with the same two points, namely, the common lines of the conics determined by t_1 and

t_2, respectively. If we ask that the tangent at t_1 be a line of its conic, it is readily seen that t_1 must be a point of inflection. For let the parameter of t_1 be zero and the tangent be the line $x_0=0$, then the conic for $t=0$ is

$$(b\xi)(f\xi)-4(c\xi)(e\xi)+3(d\xi)^2=0, \qquad (17)$$

and if $x_0=0$ is a line of (17), then d_0 must be zero, whence $x_0=0$ is a stationary line, and the roots of $C_{2,2}=0$ coincide, showing, as we have already seen, that I_4, its discriminant, contains the stationary lines. From a point τ there will be two lines to its conic, and the locus of these lines as τ varies is a curve of class 12 given by the eliminant of $C_{1,5}$ and $C^2_{2,2}$.*

If ρ^5 has an undulation, all the conics of the system touch the undulation tangent and this becomes a double line of the I_4. Hence for three undulations the linear system of conics given by the $C_{2,2}$ is the net of conics on the three undulation tangents and I_4 becomes rational.

The six cyclic lines α are the sections of ρ^5 for which the canonizant vanishes, and it has already been shown that the $C_{3,3}$ gives the linear system of cubic curves on these lines.† Now each line α has a unique apolar quadratic, and if t_1 and t_2 be the roots of such a quadratic, the $C_{3,3}$ formed for t_1 and t_2, respectively, will each have the given line α for a double line.

The $C_{3,5}$ is the Jacobian of $C_{2,2}$ and $C_{1,5}$, and for a given τ is a cubic curve. The lines from τ to this curve are the two lines to the $C_{2,2}$ and the tangent at τ. In the same way $C_{3,9}$ is another cubic curve, being the Jacobian of $C_{1,5}$ and $C_{2,6}$. The tangent at τ is a double line of this cubic with one of the contacts there. The osculant conic $C_{2,6}$ likewise has contact with ρ^5 at that point.

The pencil of lines through any point τ of ρ^5 cuts out a pencil of binary quartics, and two members of this pencil are self-apolar quartics. Now the $C_{4,4}$ which gives the quartics on the stationary lines, when formed for a given point, has contact with ρ^5 at that point and contains the two lines through it cutting out self-apolar quartics. Similarly the $C_{4,6}$ determines a quartic curve containing the lines to the corresponding $C_{3,3}$ and having contact with ρ^5 at that point. These quartics contain the lines α.

The quintic curve given by the $C_{5,1}$ and formed for a point τ will contain the lines to the $C_{3,3}$ and also the two self-apolar lines through τ; while the quintic given by $C_{5,3}$ contains the lines to the $C_{3,3}$ and the $C_{2,2}$, since it is the Jacobian of these two forms. Now the $C_{5,1}$ and the $C_{5,3}$ both vanish identically when $C_{1,5}$ contains a triple root or when $C_{1,5}$ is a cyclic quintic. Therefore:

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* Salmon, "Higher Algebra," 4th ed., p. 260.

† Conner, Johns Hopkins University Circular, February, 1911, p. 67.

The coefficients of $C_{5,1}$ and $C_{5,3}$ give the linear system of six independent quintic curves on the fifteen lines α and ε.

It is obvious that the sextic curve $C_{6,4}$ contains the five lines to the $C_{5,1}$ and the tangent at τ, since it is formed by operating with $C_{5,1}$ on $C_{1,5}$. In general when any covariant is thus formed from two other known covariants, the lines from any point τ to the particular covariant curve it determines are likewise known.

Let us now take one of the cyclic lines for a reference line, say

$$x_0=a_0t^5+f_0, \tag{18}$$

and form the $C_{3,3}$ and $C_{5,1}$ for any point τ; the terms in ξ_0^3 and ξ_0^5, respectively, in these equations do not appear, since both contain the reference line. Now by taking the linear polar of line (18) with reference to the $C_{3,3}$ we obtain the equation of the point where the curve touches the given line.* Since there is no term in ξ_0^3, to obtain the linear polar we need only consider the term in ξ_0^2, which is

$$a_0f_0\tau\{(c_1\tau+d_1)\xi_1+(c_2\tau+d_2)\xi_2\}\xi_0^2.$$

Therefore the point of contact of $C_{3,3}$ with (18) is given by

$$(c_1\tau+d_1)\xi_1+(c_2\tau+d_2)\xi_2=0. \tag{19}$$

In the same way the linear polar of (18) with reference to $C_{5,1}$ is obtained. The highest term in ξ_0 being $a_0^2f_0^2\{(c_1\tau+d_1)\xi_1+(c_2\tau+d_2)\xi_2\}\xi_0^4$, we see that the required polar is given by equation (19). Hence it follows that:

For a given τ the $C_{3,3}$ and the $C_{5,1}$ have the same contact on lines α.

These lines then represent twelve common lines of the two curves, the remaining three being the lines from τ to its $C_{3,3}$, as we have seen that these are lines of the $C_{5,1}$. By the same method it is readily seen that for a given point:

The $C_{5,3}$, $C_{5,7}$, $C_{7,1}$ and $C_{7,5}$ have the same contact on lines α; also the $C_{7,1}$ and $C_{8,2}$ have the same contact on lines ε.

§ 7. *The Invariants.*

Since all the invariants vanish identically when $C_{1,5}$ has a triple root, it follows that all the invariant curves contain the lines ε. The quartic curve I_4 contains also lines β and γ as I_4 vanishes for the canonical form of the binary quintic cut out by any of these lines. Now the line equation of ρ^5 is given by the discriminant of $C_{1,5}$, namely, $I_4^2-128I_8$. Therefore lines γ and ε represent all the lines common to I_4 and ρ^5.

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* Dually, the linear polar of any point on a given curve is the tangent to the curve at that point.

The lines ε are double lines of the curve I_8. This is easily seen by taking two stationary lines and the line joining the corresponding points of inflection for a triangle of reference. Calling the stationary lines $x_0=0$ and $x_2=0$ and the parameters of the points of inflection 0 and ∞, respectively, any line section will be given by

$$(a\xi)(at)^5 \equiv a_0\xi_0 t^5+5(b_0\xi_0+b_1\xi_1)t^4+10(c_0\xi_0+c_1\xi_1)t^3 \\ +10(d_1\xi_1+d_2\xi_2)t^2+5(e_1\xi_1+e_2\xi_2)t+f_2\xi_2. \qquad (20)$$

On forming the I_8 of this we observe that the two highest powers of ξ_0 and ξ_2 do not appear, showing that the stationary lines are double lines. In a similar manner lines γ are seen to be lines of I_8, for if

$$x_0=a_0t^5+10d_0t^2 \qquad (21)$$

be a reference line, from the explicit form of I_8[*] the highest term in ξ_0 is $-27a_0^2d_0^5(f_1\xi_1+f_2\xi_2)\xi_0^7$, and taking the linear polar of (21) with reference to the curve the equation of the point of contact is

$$f_1\xi_1+f_2\xi_2=0. \qquad (22)$$

Now taking the line equation of ρ^5[†] we see that its contact on (21) is likewise given by (22). Therefore:

The curves I_8 *and* ρ^5 *have the same contact on lines* γ.

The invariant I_{12}[‡] is the discriminant of the $C_{3,3}$ and represents a curve of class 12, α being four-fold lines and ε double lines. If we examine the relation of I_{12} to the lines ε, by means of (20), we find that I_{12} has contact with ρ^5 at its points of inflection. The nature of these contacts may be determined by considering the conjugate curve, which will be designated by $\bar{\rho}^5$. This curve has its line sections apolar to the line sections of ρ^5. In general every line section of ρ^5 will determine a $C_{1,5}$ which has a unique apolar cubic, its canonizant, and this cubic will be represented by three points on a line of $\bar{\rho}^5$. Now since I_{12} is the locus of line sections of ρ^5 for which the canonizant has a double root, it follows that to every line of I_{12} there corresponds a line of $\bar{\rho}^5$. On the other hand, since the point of contact of a line of $\bar{\rho}^5$ taken with any one of the three extra intersections will represent the canonizant which gives a line of I_{12}, we see that to every line of $\bar{\rho}^5$ there correspond three lines of I_{12}. Hence:

The curve I_{12} *is in a one-to-three correspondence with the line equation of the conjugate curve.*

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[*] Salmon, "Higher Algebra," p. 229.

[†] Salmon, "Higher Algebra," p. 230.

[‡] The explicit form is given in Cayley, Vol. II, p. 294. All the forms of $C_{1,5}$ are given in full in this volume.

Now when $C_{1,5}$ has a triple root, its canonizant has the same triple root, and to a stationary line of ρ^5 must correspond a stationary line of I_{12}. This curve then has points of inflection which coincide with the points of inflection of ρ^5, and lines ε each represent six common lines of the two curves.

I_{12} also contains the lines β, and it is readily seen that I_4 and I_{12} have the same contact on these lines. Hence the lines β and ε are all the lines common to these two curves.

The skew invariant I_{18} gives a curve of class 18 containing all the sets of lines mentioned previously except γ. From the equation of this curve one observes that lines α are five-fold lines and δ and ε are triple lines. However, the binary equation giving the lines from a point of inflection of ρ^5 to I_{18} shows that from this point the stationary line represents five lines to the I_{18}. This indicates that the curve has a point of inflection and an ordinary contact at every point of inflection of the ρ^5, whence all the common lines of I_{12} and I_{18} are given by lines α, β and ε.

The lines α, δ and ε are the base lines of the pencil of quintic curves given by the $C_{5,1}$, and, for any member of the pencil, these base lines are equivalent to 87 common lines with the I_{18}. There are then three other common lines and it is obvious that they are given by the lines from the point τ, which determines the $C_{5,1}$, to the $C_{3,3}$. Or, since I_{18} is the eliminant of $C_{5,1}$ and $C_{3,3}$, we can define this curve to be the locus of lines from a varying point τ to its $C_{3,3}$.

§ 8. *The Pencil* $I_4^2+\lambda I_8$.

It has been stated that I_4 contains lines γ and ε, and since I_8 has ε for double lines and has contact with ρ^5 on γ it is evident that $I_4^2+\lambda I_8$ forms a pencil of octavics with ε as double lines and having simple contact on γ. These two sets of lines are equivalent to 64 common lines and are the base lines of the pencil. The base curve $I_4^2-128I_8$ is a member of this pencil, and of other members the I_4 and I_8 have been mentioned. It may be added that the lines α are double lines of I_8, and this curve may be defined as the locus of line sections of ρ^5 for which the $C_{2,2}$ and $C_{6,2}$ are harmonic pairs.

Another member of this pencil is of especial interest. We found that from a point τ there are two lines whose intersections with ρ^5 are self-apolar quartics, and the locus of these two lines for a varying τ is the curve $I_4^2-3I_8$.* Lines δ contain a binary cubic and its Hessian, and it is obvious that when τ is either of the Hessian points the remaining four are self-apolar, hence the lines δ are double lines of $I_4^2-3I_8$. These with lines ε make nineteen double lines of this curve; it is therefore of genus 2 and has a pencil of adjoint curves of class 5.* Since these lines are part of the base lines of the pencil $C_{5,1}$, it follows that:

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*Salmon, "Higher Algebra," p. 259.

The pencil of adjoint quintics of the invariant curve $I_4^2—3I_8$ *is given by the covariant* $C_{5,1}$.

For a given τ the two extra lines, which are a set of the involution, are the two lines through τ containing self-apolar quartics, since if four roots of $C_{1,5}$ are self-apolar the remaining root is the covariant $C_{5,1}$. Hence the pairs of lines in the involution intersect on the base curve and are the lines by whose locus the $I_4^2—3I_8$ was defined.

The following table is given to show which covariants contain any of the special line sections and their multiplicity.

Lines	$C_{3,3}$	I_4	$C_{4,6}$	$C_{5,1}$	$C_{5,3}$	$C_{5,7}$	$C_{6,2}$	$C_{6,4}$	$C_{7,1}$	$C_{7,5}$	I_8	ρ^5	$C_{8,2}$	$C_{9,3}$	$C_{11,1}$	I_{12}	$C_{13,1}$	I_{18}
α	1	..	1	1	1	1	2	1	1	1	2	..	2	3	3	4	2	5
β	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	1
γ	..	1	..	..	..	..	..	..	..	..	1	1	..	..	..	..	..	..
δ	..	..	..	1	..	..	..	..	1	1	..	..	1	..	1	..	2	3
ε	..	1	..	1	1	..	1	1	1	1	2	2	1	1	2	2	2	3
ζ	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	1
η	..	..	..	..	..	..	..	..	..	..	..	2	..	..	..	..	..	1

§ 9. *The Discriminant of the* $C_{4,4}$.

If $C_{1,5}$ has a double root, this is also a double root of its $C_{4,4}$. Interpreted geometrically this means that for a given point τ of ρ^5 the $C_{4,4}$ will have contact there, and the discriminant of $C_{1,5}$ must be a factor of the discriminant of the $C_{4,4}$. Again, if four points of a binary quintic are self-apolar, the remaining point is a double root of the $C_{4,4}$. This is readily seen by making the one point $t=0$; then in the $C_{1,5}$ we have $f=0$, and the last two terms of the $C_{4,4}$ become

$$—(2det+e^2)(6ae—15bd+10c^2). \qquad (23)$$

Now the second factor is the condition that the four remaining points of $C_{1,5}$ be self-apolar, and hence $t=0$ is a double root of the $C_{4,4}$. We have seen that

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* Dually a curve of order n and genus 2 has a pencil of adjoint curves of order $(n—3)$ on the $\frac{1}{2}(n—1)(n—2)—2$ double points. These have two extra intersections each, and set up an involution $I^2_{1,1}$ on the base curve.

$I_4^2-3I_8$ gives the locus of lines whose intersections with ρ^5 contain self-apolar quartics, and this invariant must also be a factor of the discriminant of the $C_{4,4}$.

The discriminant is of degree 24 in the original coefficients and is found from the two invariants of the $C_{4.4}$. These are:

$$\left.\begin{aligned} 12g_2&\equiv I_4^2+72I_8, \\ 216g_3&\equiv I_4(I_4^2-378I_8). \end{aligned}\right\} \tag{24}$$

These invariants have well-known meanings, and the discriminant of which two factors have been found is

$$g_2^3-27g_3^2\equiv I_8(I_4^2-3I_8)(I_4^2-128I_8). \tag{25}$$

If we consider the $C_{1,5}$ as given by a line of γ, the $C_{4,4}$ will have a triple root at the point of contact, and it is obvious that the quartic curve formed for that point will have a point of inflection there, the line γ being a stationary line.

§ 10. *The Hermitian of the $C_{4,4}$.*

Some further relations in connection with the $C_{4,4}$ are found by means of the Hermitian of the system of curves. In general, if we ask for the locus of lines which intersect a linear system of $(n+1)$ curves of order n in sets of n points in an involution, we find the locus to be a curve of class

$$\frac{n(n+1)}{2},$$

which I will call the Hermitian of the system of n-ics.

If the curves all have a point in common, that point becomes a part of the locus, and the class of the remaining part is reduced by one. The same is true for every point which the system may have in common.

Now consider the dual statement, and take for illustration a net of conics. The Hermitian is a curve of order 3 and, in this case, is the locus of the degenerate members of the net. From any point of this locus the tangents to the conics are in an involution.* If the conics are given symbolically by $(a\xi)^2$, $(b\xi)^2$ and $(c\xi)^2$, respectively, the required locus is

$$|abx|\,|bcx|\,|cax|=0. \tag{26}$$

In general, for an n-fold infinity of n-ics given by

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* Salmon, "Conic Sections," p. 360.

$$(a_0\xi)^n,\ (a_1\xi)^n,\ \ldots .,\ (a_n\xi)^n$$

the Hermitian is

$$\prod^{\frac{n(n+1)}{2}} |a_0a_1x|\,|a_0a_2x| \ldots . |a_{n-1}a_nx| = 0, \tag{27}$$

and this involves the coefficients of each of the n-ics linearly. If the three conics have a line in common, they will be seen in an involution from any point of the line; and if the conics are a net on three lines, the Hermitian becomes the product of the three lines.

Similarly, for a triple infinity of cubic class curves the locus of points from which these appear in an involution is a curve of order 6. If the cubics touch six lines, as, for example, the cubics on lines α given by the $C_{3,3}$, then the Hermitian is the product of those lines. If the cubics have five lines in common, the given curve consists of these five lines and another line, the locus of a point which, with the conic on the five lines, is a cubic of the system.

In a system of ∞^4 quartics the required locus is a curve of order 10. Let the quartics under consideration be given by the $C_{4,4}$ of a general line section of ρ^5. This, being a system of curves on lines ε, must have these nine lines as part of the Hermitian. The remaining part must be a line such that any point of it, together with the unique cubic on the nine lines, is a degenerate quartic of the system. The required line can be identified as follows:

The $C_{4,4}$, being of degree 4 in the coefficients of ρ^5, and in the ξ's, and of order 4 in the t's, may be represented by $\{A^4,\ \xi^4,\ t^4\}$. The Hermitian is of order 10 and is given by

$$\prod^{10} |abx|\,|acx| \ldots . |dex| \equiv \{A^{20},\ x^{10}\}, \tag{28}$$

since it involves the coefficients of the five quartics linearly, and each of these is of degree 4 in the A's. Hence equation (28) must contain the lines ε as a factor, and the form of the remaining factor is easily found. The ρ^5 in point form is $\{A,\ \xi,\ t^5\}$ and in line form is $\{A^2,\ x,\ t^8\}$, while the equation giving the points of inflection is represented by $\{A^3,\ t^9\}$. Now the eliminant of $\{A^2,\ x,\ t^8\}$ and $\{A^3,\ t^9\}$, which is $\{A^{42},\ x^9\}$, gives the product of the stationary lines and an additional factor, the cusp-invariant; for if x is a point on a stationary line, t will be a common root of these two forms, while if ρ^5 has a cusp, the cusp-parameter will factor out for any point x of the plane. The cusp-invariant, then, which is of degree 24 in the A's, must be a factor of the eliminant, the remaining factor being the stationary lines, or:

$$\{A^{42},\ x^9\} \equiv \{A^{24}\} \cdot \{A^{18},\ x^9\}. \tag{29}$$

The same result may be obtained by taking the discriminant of $\{A^2,\ x,\ t^8\}$, which gives the equation of ρ^5 together with the product of the stationary lines. The discriminant is $\{A^{28}, x^{14}\}$, and since the equation of ρ^5 is $\{A^{10}, x^5\}$, the remaining factor is the same as found above.

Since the stationary lines are a part of the Hermitian of the system of quartics given by the $C_{4,4}$, we have

$$\{A^{20}, x^{10}\} \equiv \{A^{18}, x^9\} \cdot \{A^2, x\}. \tag{30}$$

The remaining part of the locus must be a covariant line, and there is only one covariant line of degree 2 in the coefficients; namely, the line whose intersection with ρ^5 is a binary quintic apolar to all line sections. Its equation is

$$|abx|\,|\alpha\beta| \equiv |afx| - 5\,|bex| + 10\,|cdx| = 0. \tag{31}$$

Therefore from any point of this line the system of quartics given by the $C_{4,4}$ *appears in an involution.*

Then there will be a set of four lines apolar to all sets of four lines in the involution, and any point of the line (31) with the cubic on the nine stationary lines is a quartic of the system given by the $C_{4,4}$.

§ 11. *A Quartic Curve for Every Point in the Plane.*

Following is a method by which a curve of the system given by the $C_{4,4}$ is associated with every point of the plane. Take any two lines ξ and η and form the $C_{4,4}$ for the pencil $\xi + \lambda\eta$; this gives a quartic in λ, each coefficient of which involves t to the fourth degree. Suppose ξ and η cut out the two binary quintics $(\alpha t)^5$ and $(\beta t)^5$, respectively. These quintics have a combinant quartic which we can write $|abx|\,|\alpha\beta|^3(\alpha t)^2\,(\beta t)^2$, x being the point of intersection of the two lines. It is seen at once that:

If x is a point of line (31), *then each coefficient of the quartic in λ is apolar to the given combinant.*

This is most easily verified by taking for a reference triangle two stationary lines and line (31). Then if x is a point of (31) the $C_{4,4}$ of $(\alpha t)^5 + \lambda(\beta t)^5$, considered as a quartic in λ, has each coefficient apolar to $|abx|\,|\alpha\beta|^3(\alpha t)^2(\beta t)^2$. Now if we operate with this combinant formed for a point x of (31) on the $C_{4,4}$ of the general line section $(a\xi)\,(\alpha t)^5$, the result is a quartic in the ξ's from which the point x factors, leaving the cubic on the stationary lines.

The combinant above is a function of $\{A^2, x, t^4\}$, while the $C_{4,4}$ of a line section has the form $\{A^4, \xi^4, t^4\}$. Now the apolarity relation of these is $\{A^6, \xi^4, x\}$, but we have seen that this form vanishes when the line ξ and point x are incident, provided x is a point of (31). In other words $\{A^6, \xi^4, x\} = 0$ when both $(\xi x) = 0$ and $\{A^2, x\} = 0$. Hence there exists the relation

$$\{A^6, \xi^4, x\} = \{A^6, \xi^3\}(\xi x) + \{A^4, \xi^4\}\{A^2, x\}, \tag{32}$$

and this indicates that the cubic on the stationary lines is of degree 6 in the coefficients; also $\{A^4, \xi^4\}$ must be the I^4, since it is the only invariant of degree 4 in the coefficients.

The I_4 and the cubic $\{A^6, \xi^3\}$ each contain the stationary lines and hence have three other common lines giving rise to an invariant triangle of ρ^5. The degree of this in the coefficients is calculated by taking the eliminant of $\{A^4, \xi^4\}$, $\{A^6, \xi^3\}$, and (ξx). The result on eliminating ξ is $\{A^{36}, x^{12}\}$, and on removing the factor which is the product of the stationary lines there remains the invariant triangle which is $\{A^{18}, x^3\}$.

From the relation (32) found above there is a double infinity of quartic curves picked out of the system given by the $C_{4,4}$. Take any point x of the plane and form the combinant quartic of the pencil of line sections on x. Then operating with this combinant on the $C_{4,4}$, the result is a quartic curve on the stationary lines. The pencil of this quartic and the I_4 contains the cubic on the stationary lines and the point x as a degenerate member. There is then a quartic curve of the system given by the $C_{4,4}$, associated uniquely with every point of the plane, and the pencil of quartics formed by this curve and the I_4 will have for base lines the stationary lines, the invariant triangle mentioned above, and the four lines from the given point to the I_4.

§ 12. *The Cubic Curve on the Stationary Lines of ρ^5.*

There are three sets of covariant points on the ρ^5 of degree 3 in the coefficients and of orders 9, 5, and 3 in the variable.* These we shall write in the forms $\{A^3, t^9\}$, $\{A^3, t^5\}$, and $\{A^3, t^3\}$, respectively. Now the $C_{1,5}$ has three fundamental forms of the third degree in the coefficients, namely, $C_{3,9}$, $C_{3,5}$, and $C_{3,3}$, which for a line section of ρ^5 are in the forms $\{A^3, \xi^3, t^9\}$, $\{A^3, \xi^3, t^5\}$, and $\{A^3, \xi^3, t^3\}$, respectively. On writing the apolarity relation of each pair of quantics having the same order in the variable t, the result in each case gives a cubic curve of degree 6 in the A's. Since the cubic on the stationary lines is in the form $\{A^6, \xi^3\}$, we shall determine whether it belongs to the system of these cubic curves.

The roots of the covariant form $\{A^3, t^9\}$ equated to zero give the parameters of the nine points of inflection, the symbolical expression for this covariant being

$$|abc|\,|\alpha\beta|\,|\beta\gamma|\,|\gamma\alpha|\,(\alpha t)^3\,(\beta t)^3\,(\gamma t)^3; \tag{33}$$

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*See Coble, "Symmetric Binary Forms and Involutions," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, p. 359.

and when the ρ^5 is given by equation (13), the developed form is

$$\begin{aligned}|abc|t^9&+3|abd|t^8+3\{2|acd|+|abe|\}t^7\\&+\{8|ace|+10|bcd|+|abf|\}t^6\\&+3\{2|ade|+5|bce|+|acf|\}t^5\\&+3\{2|bcf|+5|bde|+|adf|\}t^4\\&+\{8|bdf|+10|cde|+|aef|\}t^3\\&+3\{2|cdf|+|bef|\}t^2+3|cef|t+|def|=0.\end{aligned} \tag{34}$$

The covariant of order 5, $\{A^3, t^5\}$, gives the parameters of the five points cut out by the covariant line $|abx|\,|\alpha\beta|^5$; the symbolical expression is

$$|abc|\,|\alpha\beta|^3\,|\beta\gamma|\,|\gamma\alpha|\,(\alpha t)\,(\beta t)\,(\gamma t)^3, \tag{35}$$

and the expanded form

$$\begin{aligned}\{|abe|&-2|acd|\}t^5+\{|abf|-10|bcd|\}t^4+2\{|acf|-5|bce|\}t^3\\&+2\{|adf|-5|bde|\}t^2+\{|aef|-10|cde|\}t+\{|bef|-2|cdf|\}=0.\end{aligned} \tag{36}$$

The remaining covariant form is

$$|abc|\,|\alpha\beta|^3\,|\beta\gamma|^2\,|\gamma\alpha|\,(\alpha t)\,(\gamma t)^2, \tag{37}$$

of which the expanded form is found to be

$$\begin{aligned}\{2|abf|&-5|ace|+20|bcd|\}t^3+3\{|acf|-5|ade|+5|bce|\}t^2\\&-3\{|adf|-5|bcf|+5|bde|\}t-\{2|aef|-5|bdf|+20|cde|\}=0.\end{aligned} \tag{38}$$

Now take the reference triangle so that any line section $C_{1,5}$ is given by equation (20) and form the $C_{3,9}$, $C_{3,5}$, and $C_{3,3}$, respectively, of this equation. Let the apolarity relation of (34) and $C_{3,9}$ be denoted by C_1, that of (36) and $C_{3,5}$ by C_2, and of (38) and $C_{3,3}$ by C_3. These are the three cubic curves each involving the original coefficients to the sixth degree; and since $x_0=0$ is a stationary line, it is only necessary to find a covariant curve which does not contain the term in ξ_0^3, as such a curve will contain all the stationary lines. We observe then if there is a member of the system

$$\lambda C_1+\mu C_2+\nu C_3=0 \tag{39}$$

which is free from the term in ξ_0^3. Obtaining C_1, C_2, and C_3, it is easily seen that the required curve is a linear combination of these three curves, the numerical coefficients being readily determined, and hence:

$$C\equiv 84C_1-35C_2-80C_3=0 \tag{40}$$

is the cubic line curve on the stationary lines of the rational quintic.